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# On the commuting charges for the highest dimension SU(2) operators in planar $\mathcal{N} = 4$ SYM

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Abstract: We consider the highest anomalous dimension operator in the SU(2) sector of planar $\mathcal{N} = 4$ SYM at all-loop, though neglecting wrapping contributions. In any case, the latter enter the loop expansion only after a precise length-depending order. In the thermodynamic limit we write both a linear integral equation for the Bethe root density and a linear system obeyed by the commuting charges. Consequently, we determine the leading strong coupling contribution to the density and from this an approximation to the leading and sub-leading terms of any charge $Q_r$: it scales as $\lambda^{1/4-r/2}$, which generalises the Gubser-Klebanov-Polyakov energy law. In the end, we briefly extend these considerations to finite lengths and 'excited' operators by using the idea of a non-linear integral equation.

Keywords: Bethe Ansatz, AdS-CFT Correspondence, Lattice Integrable Models.

## Contents



## 1. Introduction

According to the AdS/CFT correspondence [1], string theory on the curved space-time $AdS_5 \times S^5$ is equivalent to a conformal quantum field theory on its boundary. In particular, type IIB superstring theory should be dual to $\mathcal{N} = 4$ Super Yang-Mills theory (SYM). The AdS/CFT correspondence is a general dictionary which would in particular equate energies of string states to anomalous dimensions of local gauge invariant operators of the quantum field theory.

One of the most important recent development in this context is the discovery of integrability in both planar field theory and string theory. In few words, integrable models appear as Bethe equations to be satisfied by 'rapidities' which parametrise on the one side composite operators (and their anomalous dimensions in SYM) and on the other side the corresponding dual objects in string theory, i.e. states (and their energies, respectively).

In this context, the basic initial result was the identification [2] of the one-loop dilatation operator of scalar gauge-invariant fields (of bare dimension $L$) with an SO(6) integrable hamiltonian with ($L$ sites). Thanks to this discovery, one could start to use the powerful technique of the Bethe Ansatz in order to compute anomalous dimensions of long operators, giving incredible boost towards a proof of the AdS/CFT correspondence. Soon afterwards, integrability at higher loops [3, 4] was discovered and an all-loops Bethe Ansatz proposed [5]. The restriction to the SU(2) sector of these Bethe equations gives the previously hinted BDS model [6], which, remarkably, was shown to be, in a precise sense, the asymptotic limit of the one-dimensional Hubbard model [7, 8].

In a parallel way, integrability in superstring theory was discovered first at classical level [9] and work at (semi)classical level allowed to interpret the complex curve subtending the dynamics in terms of density integral equations, remembering those in Bethe Ansatz theory [10]. Moving from here, the first steps towards extending integrability at quantum level showed the appearance of long-range Bethe Ansatz equations with, however, an additional dressing phase [11]. More recently, this phase has been understood as a necessary factor to guarantee the (non-relativistic) crossing symmetry of the S-matrix [12]. Eventually, a phase factor — determined by string loop expansion at first orders [13] and crossing symmetry — was proposed in [14, 15]. In [14] the complete asymptotic strong coupling (i.e. string loop) expansion of the dressing factor was given, whereas in [15] an expression valid for all values of the coupling constant was fixed.

In this paper we want to address some of the issues related to the dressing factor, in a sector, SU(2), different from that, SL(2), in which it was initially proposed [15]. First of all, we will give a proof that the factor given as a meromorphic function of the coupling constant in [15] has the asymptotic expansion proposed by [14].[1] Afterwards, we will write the thermodynamic (i.e. in the $L \to \infty$ limit) linear integral equation satisfied by the density of roots describing the highest (or, better, 'anti-ferromagnetic') anomalous dimension state and then the system of linear equations obeyed by (the eigenvalues of) the commuting charges: we also point out some differences and difficulties introduced by the presence of the dressing factor. Then, we find the leading solution to the density equation in the strong coupling limit and derive, upon integrating this solution, an approximation to the leading term of any charge. As useful exercise, we show that in a particular case, corresponding to doubling the 'physical' dressing factor, the aforementioned equations (for both the density and the charges) allow us for simple solutions in the strong coupling limit. Therefore, we are able to give the *exact* leading and the sub-leading terms of all the conserved quantities and we can understand why these contributions are *exact* in this case whereas an approximation in the 'physical' case. Eventually, we deduce an extension for finite length, $L$, of the density equation and of the charges equations by using the idea of the non-linear integral equation [16] along the lines of [17]: now the loop expansion can be trusted up to order $\lambda^{L-1}$. For simplicity's sake we will limit our attention to the anti-ferromagnetic state and *hole* type excitations on it, even because in other sectors (cf. the treatment of [15]) these are the only possible states.

## 2. Bethe equations in the SU(2) sector

It is nowadays clear that the *asymptotic* Bethe Ansatz type equations describing planar $\mathcal{N} = 4$ SYM should contain — with respect to the first proposals — a universal dressing phase [11, 14, 15]. In the SU(2) scalar sector, the BDS equations [6] ought to be modified into

$$\left[\frac{X(u_k + \frac{i}{2})}{X(u_k - \frac{i}{2})}\right]^L = \prod_{\substack{j=1 \\ j \neq k}}^{M} \frac{u_k - u_j + i}{u_k - u_j - i} \exp[2i\theta(u_k, u_j)] \,, \tag{2.1}$$

[1] Indeed, this was not completely proved in [15]: we thank M. Staudacher to have encouraged us on this matter.

with the usual notation

$$X(x) = \frac{x}{2}\left(1+\sqrt{1-\frac{\lambda}{4\pi^2x^2}}\right). \tag{2.2}$$

On the other hand, the (renormalised) dimension corresponding to the operator/solution $\{u_k\}_{k=1,\ldots,M}$ of (2.1) is formally unchanged, i.e.

$$\Delta = L + \sum_{k=1}^{M}\left[\left(\frac{1}{2}-iu_k\right)\sqrt{1+\frac{4g^2}{\left(\frac{1}{2}-iu_k\right)^2}}+\left(\frac{1}{2}+iu_k\right)\sqrt{1+\frac{4g^2}{\left(\frac{1}{2}+iu_k\right)^2}}-1\right], \tag{2.3}$$

where we have introduced $\lambda = Ng_{\mathrm{YM}}^2 = 16\pi^2g^2$, the 't Hooft coupling of planar theory ($N\to\infty, g_{\mathrm{YM}}\to 0$). The term *asymptotic* means exactly that this Ansatz is believed to give the exact loop expansion to the anomalous dimension up to *wrapping corrections*, starting at order $\lambda^L$. The complete phase factor has been conjectured to be the $\kappa=1$ of the following double series [15]

$$\theta(u_k,u_j) = \kappa\sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty}\beta_{r,r+1+2\nu}(g)[q_r(u_k)q_{r+1+2\nu}(u_j)-q_r(u_j)q_{r+1+2\nu}(u_k)]. \tag{2.4}$$

However, we prefer to keep $\kappa$ unfixed, because in section 4 we will study the case $\kappa = 2$, which mathematically reveals surprising simplifications. In order to fix the notations in (2.4), we remind that $q_r(u)$ is the magnon, $u$, $r$-th charge[2]

$$q_r(u) = \left(\frac{8\pi^2}{\lambda}\right)^{r-1}\frac{1}{i^{r-2}(r-1)}\left\{\left[\left(\frac{1}{2}-iu\right)\sqrt{1+\frac{4g^2}{\left(\frac{1}{2}-iu\right)^2}}-\left(\frac{1}{2}-iu\right)\right]^{r-1}+\right.$$
$$\left.+(-1)^r\left[\left(\frac{1}{2}+iu\right)\sqrt{1+\frac{4g^2}{\left(\frac{1}{2}+iu\right)^2}}-\left(\frac{1}{2}+iu\right)\right]^{r-1}\right\}, \tag{2.5}$$

and the function $\beta_{r,r+1+2\nu}(g)$ are meromorphic functions of $g$, introduced and studied in [15]. In this paper their weak coupling expansion was proposed as

$$\beta_{r,r+1+2\nu}(g) = \sum_{\mu=\nu}^{\infty} g^{2r+2\nu+2\mu}\beta_{r,r+1+2\nu}^{(r+\nu+\mu)}, \tag{2.6}$$

the coefficients $\beta_{r,r+1+2\nu}^{(r+\nu+\mu)}$ being

$$\beta_{r,r+1+2\nu}^{(r+\nu+\mu)} = 2(-1)^{r+\mu+1}\frac{(r-1)(r+2\nu)}{2\mu+1}\binom{2\mu+1}{\mu-r-\nu+1}\binom{2\mu+1}{\mu-\nu}\zeta(2\mu+1). \tag{2.7}$$

These weak-coupling Taylor series around $g=0$ have finite radius of convergence and define unambiguously these functions of $g$. In particular, they can be written in terms of integrals as follows:

$$\beta_{r,r+1+2\nu}(g) = 2(r-1)(r+2\nu)(-1)^{\nu}g^{2r+2\nu-1}\int_0^{\infty}dt\frac{J_{r-1}(2gt)J_{r+2\nu}(2gt)}{t(e^t-1)}. \tag{2.8}$$

[2] The following expression may be interpreted as valid even for the momentum, provided in the sense of the limit $r\to 1$.

Indeed, by expanding the product of Bessel functions in the integrand of (2.8) in Taylor series around the origin [18], one can easily check the expansion (2.6).

Now we may give a proof that the functions defined by (2.8) enjoy at strong coupling ($g \to +\infty$) the asymptotic expansion proposed in [14]. We reparametrise the $\beta$s as

$$\begin{aligned} \beta_{r,s}(g) &= g^{r+s-2} c_{r,s}(g)\,, \\ c_{r,s}(g) &= 2\cos\left[\frac{\pi(s-r-1)}{2}\right](r-1)(s-1)\int_0^\infty dt \frac{J_{r-1}(2gt)J_{s-1}(2gt)}{t(e^t-1)}\,, \end{aligned} \tag{2.9}$$

in order to make contact with the notations of [14]. From their definition it is always $r \geq 2$, $s \geq r+1$ and their difference $s-r$ equals a positive odd integer. We now perform the change of variables $t \to t/2g$ and expand the function $1/(e^{t/2g}-1)$ in powers of $t/2g$. Then we formally exchange this power series with the integral, obtaining

$$\int_0^\infty dt \frac{J_{r-1}(2gt)J_{s-1}(2gt)}{t(e^t-1)} = \sum_{n=0}^{\infty} \frac{1}{(2g)^{n-1}} \frac{B_n}{n!} \int_0^\infty dt J_{r-1}(t) J_{s-1}(t) t^{n-2}\,, \tag{2.10}$$

with $B_n$ the Bernoulli numbers. The series in (2.10) is for now formal, since the integral is defined only for $n \leq 2$. However, the idea is to extend the result obtained for general $n$, $0 \leq n \leq 2$, to arbitrary values of $n$. This will eventually define the asymptotic expansion of the coefficients $c_{r,s}(g)$. Indeed, we may specialise formula 6.574.2 of [18] to the form

$$\int_0^\infty dt J_{r-1}(t) J_{s-1}(t) t^{n-2} = \frac{2^{n-2}\Gamma(-n+2)\Gamma\left(\frac{r+s-3+n}{2}\right)}{\Gamma\left(\frac{s-r+3-n}{2}\right)\Gamma\left(\frac{r+s+1-n}{2}\right)\Gamma\left(\frac{r-s+3-n}{2}\right)}\,. \tag{2.11}$$

Strictly speaking this formula is valid for $n = 0, 1$. When $n = 0$ the last Gamma function in the denominator is divergent unless $s = r+1$, and then

$$\int_0^\infty dt J_{r-1}(t) J_{s-1}(t) t^{-2} = \delta_{s,r+1} \frac{1}{4r(r-1)}\,. \tag{2.12}$$

For $n = 1$ the expression may be simplified into

$$\int_0^\infty dt J_{r-1}(t) J_{s-1}(t) t^{-1} = \frac{2\sin\left(\pi\frac{s-r}{2}\right)}{\pi} \frac{1}{(s+r-2)(s-r)}\,. \tag{2.13}$$

For what concerns the values $n \geq 2$, we remark that the right hand side of (2.11) can be well defined not only for $n = 2$, but also for all even $n$. In order to prove this, it is useful to introduce the integer $k \geq 0$, such that $s-r = 2k+1$, and the regularisator $\delta \to 2$

$$\int_0^\infty dt J_{r-1}(t) J_{s-1}(t) t^{n-\delta} = \frac{2^{n-\delta}\Gamma(-n+\delta)\Gamma\left(k+r+\frac{n-\delta}{2}\right)}{\Gamma\left(k+1-\frac{n-\delta}{2}\right)\Gamma\left(k+r-\frac{n-\delta}{2}\right)\Gamma\left(-k-\frac{n-\delta}{2}\right)}. \tag{2.14}$$

In this limit, the first Gamma function in the numerator as well as the last in the denominator diverge, but their ratio stays finite:

$$\lim_{\delta\to 2} \frac{\Gamma(\delta-n)}{\Gamma\left(-k+\frac{\delta-n}{2}\right)} = \frac{1}{2}(-1)^{\frac{n-s+r-1}{2}} \frac{\Gamma\left(\frac{s-r-1+n}{2}\right)}{\Gamma(n-1)}\,. \tag{2.15}$$

Therefore, we may write

$$\int_0^\infty dt J_{r-1}(t) J_{s-1}(t) t^{n-2} = (-1)^{\frac{n-s+r-1}{2}} \frac{2^{n-3}\Gamma\left(\frac{s-r-1+n}{2}\right)\Gamma\left(\frac{s+r-3+n}{2}\right)}{\Gamma(n-1)\Gamma\left(\frac{s-r+3-n}{2}\right)\Gamma\left(\frac{s+r+1-n}{2}\right)}, \tag{2.16}$$

for all $n$ even, $n \geq 2$. Because of the divergence of the second Gamma function in the denominator, the above result is different from zero only if

$$n \leqslant 2k+2 = s-r+1\,. \tag{2.17}$$

We now plug (2.16) into (2.10) and obtain the coefficients $c_{r,s}^{(n)}$ of the asymptotic expansion

$$c_{r,s}(g) = \sum_{n=0}^{\infty} c_{r,s}^{(n)} g^{1-n}\,, \tag{2.18}$$

in the form

$$c_{r,s}^{(0)} = \delta_{r+1,s}\,, \quad c_{r,s}^{(1)} = -\frac{2}{\pi}\frac{(r-1)(s-1)}{(s+r-2)(s-r)}\,. \tag{2.19}$$

and for $n \geq 2$

$$c_{r,s}^{(n)} = \frac{1}{(-2\pi)^n \Gamma(n-1)} \zeta(n)(r-1)(s-1) \frac{\Gamma\left(\frac{s+r+n-3}{2}\right)\Gamma\left(\frac{s-r+n-1}{2}\right)}{\Gamma\left(\frac{s+r-n+1}{2}\right)\Gamma\left(\frac{s-r-n+3}{2}\right)}\,. \tag{2.20}$$

In fact, this expression for any integer $n \geq 2$ may be obtained upon expressing the Bernoulli number $B_n$ via the Riemann zeta function $\zeta(n)$, even for odd $n$, as[3]

$$B_n = (-1)^{\frac{n}{2}-1} \frac{\zeta(n) n!}{2^{n-1}\pi^n}\,. \tag{2.21}$$

To summarise, we have obtained the strong coupling expansion (2.18), (2.20) for the functions $c_{r,s}(g)$ entering the dressing factor. Notice that in formulæ (2.19), (2.20) $r \geq 2$ and $s-r$ equals a positive odd integer and (2.20) implies when $n$ is even $c_{r,s}^{(n)} = 0$ for $n > s-r+1$.

## 3. Thermodynamic limit of the highest energy state

In the infinite length limit $L \to \infty$, there is no possibility for wrapping and consequently the equations (2.1) should give an exact description of the SU(2) scalar sector. In this limit, the Bethe roots become closer and closer in such a way to form a continuous distribution. The latter may be described by a density function $\rho(u)$ and the Bethe equations turn into a linear integral equation.

In order to find this equation, we rewrite equations (2.1) in the logarithmic form,

$$iL \ln \frac{X\left(u_k + \frac{i}{2}\right)}{X\left(u_k - \frac{i}{2}\right)} = i \sum_{\substack{j=1 \\ j \neq k}}^{M} \ln \frac{u_k - u_j + i}{u_k - u_j - i} - 2 \sum_{\substack{j=1 \\ j \neq k}}^{M} \theta(u_k, u_j) + 2\pi K_k\,, \tag{3.1}$$

[3] It may be proved that this amounts to a further regularisation of the coefficients of the series (2.10).

where $K_k$ are integers which depend on the state we are considering. Now, for simplicity's sake we specialise our analysis to the state with uniquely $M = L/2$ real roots (no complex roots). This corresponds to the anti-ferromagnetic or highest anomalous dimension configuration and there is not even room for holes and then $K_k = k$. When $L \to \infty$ all the sums over Bethe roots $u = u_k$ may be replaced by integrals with Stieltjes measure

$$du\rho(u) = du \frac{1}{L}\frac{dk}{du}\,, \tag{3.2}$$

and hence, upon derivating with respect to $u$, each sum in (3.1) may be expressed in terms of $\rho$ itself

$$i\frac{X'\left(u+\frac{i}{2}\right)}{X\left(u+\frac{i}{2}\right)} - i\frac{X'\left(u-\frac{i}{2}\right)}{X\left(u-\frac{i}{2}\right)} = i\int_{-\infty}^{\infty} dv\rho(v)\left[\frac{1}{u-v+i} - \frac{1}{u-v-i}\right] - \\ -2\int_{-\infty}^{\infty} dv\rho(v)\frac{d}{du}\theta(u,v) + 2\pi\rho(u)\,.$$

Inserting the form (2.4) for the dressing phase, we have

$$\frac{i}{2}\left[\frac{1}{\sqrt{(u+\frac{i}{2})^2-4g^2}} - \frac{1}{\sqrt{(u-\frac{i}{2})^2-4g^2}}\right] = \pi\rho(u) + \int_{-\infty}^{\infty} dv \frac{\rho(v)}{(u-v)^2+1} - \\ -\kappa\sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty}\beta_{r,r+1+2\nu}(g)q_r'(u)\int_{-\infty}^{\infty} dv\rho(v)q_{r+1+2\nu}(v) + \\ +\kappa\sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty}\beta_{r,r+1+2\nu}(g)q_{r+1+2\nu}'(u)\int_{-\infty}^{\infty} dv\rho(v)q_r(v)\,.$$

In order to have shorter expressions, we introduce the (total) charges, normalised by a factor $1/L$,

$$Q_r = \int_{-\infty}^{\infty} dv\rho(v)q_r(v)\,. \tag{3.3}$$

As usual, we pass to the Fourier transform, using the simple expression [18]

$$\hat{q}_r(k) = 2^{r-1}\frac{(2\pi)^r}{(\sqrt{\lambda})^{r-1}}\frac{1}{i^{r-2}}\frac{J_{r-1}\left(\frac{\sqrt{\lambda}}{2\pi}k\right)}{ke^{\frac{|k|}{2}}}\,, \quad r \geq 1\,. \tag{3.4}$$

Then, we obtain for the Fourier transform of the density

$$\pi e^{-\frac{|k|}{2}}J_0\left(\frac{\sqrt{\lambda}}{2\pi}k\right) = \pi\hat{\rho}(k) + \pi\hat{\rho}(k)e^{-|k|} - \\ -\kappa\sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty}\beta_{r,r+1+2\nu}(g)2^{r-1}\frac{(2\pi)^r}{(\sqrt{\lambda})^{r-1}}\frac{1}{i^{r-3}}e^{-\frac{|k|}{2}}J_{r-1}\left(\frac{\sqrt{\lambda}}{2\pi}k\right)Q_{r+1+2\nu} + \\ +\kappa\sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty}\beta_{r,r+1+2\nu}(g)2^{r+2\nu}\frac{(2\pi)^{r+1+2\nu}}{(\sqrt{\lambda})^{r+2\nu}}\frac{1}{i^{r-2+2\nu}}e^{-\frac{|k|}{2}}J_{r+2\nu}\left(\frac{\sqrt{\lambda}}{2\pi}k\right)Q_r\,.$$



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Collecting the terms containing $\hat{\rho}(k)$ all together, we may deduce a linear integral equation

$$\hat{\rho}(k) = \frac{J_0\left(\frac{\sqrt{\lambda}}{2\pi}k\right)}{2\cosh\frac{k}{2}} + \kappa\sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty}\beta_{r,r+1+2\nu}(g)2^{r-1}\frac{(2\pi)^r}{(\sqrt{\lambda})^{r-1}}\frac{1}{i^{r-3}}\frac{J_{r-1}\left(\frac{\sqrt{\lambda}}{2\pi}k\right)}{2\pi\cosh\frac{k}{2}}Q_{r+1+2\nu} -$$
$$-\kappa\sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty}\beta_{r,r+1+2\nu}(g)2^{r+2\nu}\frac{(2\pi)^{r+1+2\nu}}{(\sqrt{\lambda})^{r+2\nu}}\frac{1}{i^{r-2+2\nu}}\frac{J_{r+2\nu}\left(\frac{\sqrt{\lambda}}{2\pi}k\right)}{2\pi\cosh\frac{k}{2}}Q_r\,. \tag{3.5}$$

Of course, this equation may entail an analogous linear equation for the density of roots $\rho(u)$. But more importantly it tells us that the charges $Q_r = Q_r(g)$ determine $\hat{\rho}(k)$ for any $g$; and on its turn $\hat{\rho}(k)$ yields the charges (3.3) as

$$Q_r = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\hat{\rho}(k)\hat{q}_r(-k) = \int_{-\infty}^{\infty}dk\hat{\rho}(k)\frac{i^{r-2}}{g^{r-1}}\frac{J_{r-1}(2gk)}{ke^{\frac{|k|}{2}}}\,. \tag{3.6}$$

Thanks to this sort of rôle exchange, we think that the form of the integral equation in terms of the charges is of particular interest. As an example of this utility, we will see in the following the derivation of a linear system of algebraic equations for the charges.

In fact, upon inserting (3.6) in (3.5), we may rewrite the equation for $\hat{\rho}(k)$ in a more explicit way

$$\hat{\rho}(k) = \frac{J_0(2gk)}{2\cosh\frac{k}{2}} + \kappa\sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty}\beta_{r,r+1+2\nu}(g)(-1)^{1+\nu}g^{1-2\nu-2r}\left[\frac{J_{r-1}(2gk)}{\cosh\frac{k}{2}}\cdot\right.$$
$$\left.\cdot\int_{-\infty}^{\infty}dp\hat{\rho}(p)\frac{J_{r+2\nu}(2gp)}{pe^{\frac{|p|}{2}}} + \frac{J_{r+2\nu}(2gk)}{\cosh\frac{k}{2}}\int_{-\infty}^{\infty}dp\hat{\rho}(p)\frac{J_{r-1}(2gp)}{pe^{\frac{|p|}{2}}}\right], \tag{3.7}$$

still slightly simplified if in terms of the $c_{r,s}(g)$ (using (2.9)):

$$\hat{\rho}(k) = \frac{J_0(2gk)}{2\cosh\frac{k}{2}} + \kappa\sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty}c_{r,r+1+2\nu}(g)(-1)^{1+\nu}\left[\frac{J_{r-1}(2gk)}{\cosh\frac{k}{2}}\cdot\right.$$
$$\left.\cdot\int_{-\infty}^{\infty}dp\hat{\rho}(p)\frac{J_{r+2\nu}(2gp)}{pe^{\frac{|p|}{2}}} + \frac{J_{r+2\nu}(2gk)}{\cosh\frac{k}{2}}\int_{-\infty}^{\infty}dp\hat{\rho}(p)\frac{J_{r-1}(2gp)}{pe^{\frac{|p|}{2}}}\right]. \tag{3.8}$$

In the end, upon highlighting in this equation, via (3.6), the explicit dependence on the conserved charges, we may derive immediately an infinite set of linear equations for them:

$$Q_s = \frac{i^{s-2}}{g^{s-1}}\left[\int_{-\infty}^{\infty}dk\frac{J_{s-1}(2gk)J_0(2gk)}{k(e^{|k|}+1)} + \right.$$
$$+2\kappa\sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty}c_{r,r+1+2\nu}(g)(-1)^{1+\nu}\int_{-\infty}^{\infty}dk\frac{J_{s-1}(2gk)J_{r-1}(2gk)}{k(e^{|k|}+1)}\frac{g^{r+2\nu}}{i^{r+2\nu-1}}Q_{r+2\nu+1}$$
$$\left.+2\kappa\sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty}c_{r,r+1+2\nu}(g)(-1)^{1+\nu}\int_{-\infty}^{\infty}dk\frac{J_{s-1}(2gk)J_{r+2\nu}(2gk)}{k(e^{|k|}+1)}\frac{g^{r-1}}{i^{r-2}}Q_r\right]. \tag{3.9}$$

We want to remark that this equations form a system of linear algebraic equations for (the eigenvalues of) the commuting charges on the highest energy state. We believe it could furnish important information on these eigenvalues and could introduce their disentanglement under particular conditions (for instance in the strong coupling regime).

We have to remark that an equivalent equation for $\hat{\rho}(k)$ has been already published in [19] by making use of their formalism of *magic kernels* [15].

Both equations (3.8) and (3.9) characterise the $L \to \infty$ limit of the highest anomalous dimension state of the scalar sector of $\mathcal{N} = 4$ SYM, once one supposes this to be described by the Ansatz (2.1). One has to remark the important modifications due to the presence of the dressing phase. Without that phase (i.e. in the framework of the BDS Bethe Ansatz), the thermodynamic expressions for $\hat{\rho}(k)$ and $Q_s$ reduce to

$$\begin{aligned}\hat{\rho}(k) &= \frac{J_0(2gk)}{2\cosh\frac{k}{2}}\,,\\ Q_s &= \frac{i^{s-2}}{g^{s-1}}\int_{-\infty}^{\infty} dk \frac{J_{s-1}(2gk)J_0(2gk)}{k(e^{|k|}+1)}\,. \end{aligned} \tag{3.10}$$

By construction, at weak coupling the dressing factor starts at order $g^6$ and therefore, up to that order, the solutions of (3.8) and (3.9) are given by (3.10). On the other hand, the strong coupling limit is dominated by contributions coming from the dressing factor and it is of great importance, since it makes contact with semiclassical results in string theory (cf. second reference of [1]).

We remark the presence in equations (3.9) of the quantities

$$\tilde{c}_{r,s}(g) = \int_0^{\infty} dt \frac{J_{r-1}(2gt)J_{s-1}(2gt)}{t(e^t+1)}\,, \tag{3.11}$$

which look very similar to the functions $c_{r,s}(g)$ appearing in the dressing factor. Actually, the elementary identity

$$\frac{1}{e^x-1} = \frac{1}{2}\left(\frac{1}{e^{\frac{x}{2}}-1} - \frac{1}{e^{\frac{x}{2}}+1}\right) \tag{3.12}$$

allows us to write, when $r \geq 2$ and $s-r$ equals a positive odd integer,

$$2\tilde{c}_{r,s}(g) = \frac{c_{r,s}(g) - 2c_{r,s}\left(\frac{g}{2}\right)}{\cos\left[\frac{\pi(s-r-1)}{2}\right](r-1)(s-1)}\,. \tag{3.13}$$

This entails the strong coupling expansion

$$\tilde{c}_{r,s}(g) = \sum_{n=0}^{\infty} \tilde{c}_{r,s}^{(n)} g^{1-n} \tag{3.14}$$

of the integrals contained in (3.9) in terms of the coefficients (2.19), (2.20):

$$\tilde{c}_{r,s}^{(n)} = \frac{(1-2^n)c_{r,s}^{(n)}}{2(r-1)(s-1)\sin\frac{\pi(s-r)}{2}}\,. \tag{3.15}$$

In particular, we get $\tilde{c}_{r,s}^{(0)} = 0$ and

$$\tilde{c}_{r,s}^{(1)} = \frac{1}{\pi}\frac{1}{(s+r-2)(s-r)}\frac{1}{\sin\frac{\pi(s-r)}{2}}\,; \tag{3.16}$$

$$\tilde{c}_{r,s}^{(n)} = \frac{1}{(-2\pi)^n\Gamma(n-1)}\zeta(n)\frac{(1-2^n)}{2\sin\frac{\pi(s-r)}{2}}\frac{\Gamma\left(\frac{s+r+n-3}{2}\right)\Gamma\left(\frac{s-r+n-1}{2}\right)}{\Gamma\left(\frac{s+r-n+1}{2}\right)\Gamma\left(\frac{s-r-n+3}{2}\right)}\,. \tag{3.17}$$

We remember that results (3.16), (3.17) are obtained when $r \geq 2$ and $s - r$ is equal to a positive odd integer. They can be extended to other values of $r, s$ by using the symmetry property $\tilde{c}_{r,s}(g) = \tilde{c}_{s,r}(g)$. Actually, this is what we will need for applications in this paper.

As a final remark, we observe that, with the before-stated restrictions on $r$ and $s$, relation (3.13) may be inverted as

$$c_{r,s}(g) = 2\cos\left[\frac{\pi(s-r-1)}{2}\right](r-1)(s-1)\sum_{n=0}^{\infty} 2^n \tilde{c}_{r,s}(2^{-n}g)\,. \qquad (3.18)$$

## 4. Strong coupling limit

It is of great interest to find the solutions of the linear equations for $\hat{\rho}$ and the charges $Q_r$ in the strong coupling limit (i.e. $g \to +\infty$), as these ought to match the semiclassical string theory expansion (provided no attention is payed to the order of the two limits $L \to \infty$ and $g \to +\infty$ as shown in [8]). Referring to formula (2.4), we will first concentrate on the case of generic $\kappa$, where we will be able to find the leading term for $\rho$ and, in a sense we will specify afterwards, to provide information on the leading term of the charges $Q_r$. In a second moment, we will consider the particular case $\kappa = 2$, where, because of considerable mathematical simplifications, it is possible to find exactly the leading terms of the density $\rho$ and of the eigenvalues of all conserved charges $Q_r$.

We have to remark that in the final proposal of [15] the value of interest is $\kappa = 1$. In this case, numerical computation of the strong coupling limit of the anomalous dimension was first performed in [20] and, subsequently, analytical confirmation of their finding was given in [21], following, though, a way different from ours. Nonetheless, we would like here to analyse the strong coupling expansion of $\hat{\rho}$ and of the charges, and the difficulties in their comparison.

### 4.1 Generic $\kappa$

Equation (3.5) clearly suggests and gives a full meaning to what was proposed in [22], i.e. that $2\cosh\frac{k}{2}\,\hat{\rho}(k)$ may be developed in series of Bessel functions

$$2\cosh\frac{k}{2}\,\hat{\rho}(k) = \sum_{m=0}^{\infty} a_{2m}(g) J_{2m}(2gk)\,. \qquad (4.1)$$

We insert this expansion in (3.8) and re-write (3.8) as a linear system of equations for the coefficients $a_{2n}(g)$:[4]

$$\begin{aligned} a_0(g) &= 1 \\ a_{2n}(g) &= 4\kappa\Bigg[\sum_{\nu=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{1+\nu}c_{2n+1,2n+2\nu+2}(g)\tilde{c}_{2m+1,2n+2\nu+2}(g)\,a_{2m}(g) + \qquad (4.2)\\ &\quad +\sum_{\nu=0}^{n-1}\sum_{m=0}^{\infty}(-1)^{1+\nu}c_{2n-2\nu,2n+1}(g)\tilde{c}_{2n-2\nu,2m+1}(g)\,a_{2m}(g)\Bigg]\,, \quad n \geq 1. \end{aligned}$$

[4] We are supposing that it is possible to exchange the symbol of integral with that of series: in general, we have no specific guarantee of the validity of this hypothesis. On the contrary, counterexamples are known; however, it is customary (cf. for instance [22]) to operate the exchange at this stage.



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We take now the limit $g \to +\infty$. Supposing that, for $n \geq 1$,

$$\lim_{g \to +\infty} a_{2n}(g) = a_{2n}^{(0)} , \tag{4.3}$$

the leading order $O(g)$ of the second of equations (4.2) is[5]

$$\tilde{c}_{1,2n+2}^{(1)} + \tilde{c}_{2n,1}^{(1)} + \sum_{m=1}^{\infty} [\tilde{c}_{2m+1,2n+2}^{(1)} + \tilde{c}_{2n,2m+1}^{(1)}]\, a_{2m}^{(0)} = 0 , \tag{4.4}$$

where $\tilde{c}_{r,s}^{(1)}$ is given by (3.16). The solution of this equation (remember that $a_0^{(0)} = 1$) is

$$a_{2m}^{(0)} = 2(-1)^m , \quad m \geq 1 . \tag{4.5}$$

Therefore, for the leading term of the density,

$$\hat{\rho}^{(0)}(k) = \frac{1}{2\cosh\frac{k}{2}} \sum_{n=0}^{\infty} a_{2n}^{(0)} J_{2n}(2gk) , \tag{4.6}$$

we obtain through summation formulæ for Bessel functions in [18], the expression

$$\hat{\rho}^{(0)}(k) = \frac{\cos(2gk)}{2\cosh\frac{k}{2}} \ \Rightarrow \rho^{(0)}(u) = \frac{1}{4}\left[\frac{1}{\cosh\pi(u+2g)} + \frac{1}{\cosh\pi(u-2g)}\right] . \tag{4.7}$$

This form for the leading density is not surprising. Numerical considerations, presented in [20] for $\kappa = 1$, show that the momentum of one *magnon* in the anti-ferromagnetic state scales, at strong coupling, as

$$p_k = \alpha_k \lambda^{-\frac{1}{4}} + \cdots . . \tag{4.8}$$

This implies that the rapidity $u_k = \frac{1}{2}\mathrm{cotg}\frac{p_k}{2}\sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p_k}{2}}$ behaves like

$$u_k = \frac{\mathrm{sgn}(\alpha_k)}{2\pi}\lambda^{\frac{1}{2}} + \cdots . \tag{4.9}$$

Therefore, Bethe roots accumulate at the two values $\pm\frac{\lambda^{\frac{1}{2}}}{2\pi} = \pm 2g$, exactly as suggested by (4.7).

Yet, owing to the presence of $g$ in the argument of the Bessel functions in the series (4.6), the leading expression for $\hat{\rho}(k)$ (4.7) can be really trusted only for $gk \simeq 1$ (and $g \gg 1$). In particular, this means that, if we insert (4.7) in (3.3), we only obtain an approximation for the leading value of (the eigenvalues of) the charges $Q_r$. For even $r$ (if $r$ is odd, we have $Q_r = 0$, since $\rho(u) = \rho(-u)$) we may indeed conclude

$$Q_r \simeq \frac{2^{2-r}}{4i^{r-2}(r-1)g^{2r-2}} \left\{ \int_{-\infty}^{\infty} \frac{du}{\cosh\pi u} \left[ \left(\frac{1}{2} - iu + 2ig\right) \sqrt{1 + \frac{4g^2}{\left(\frac{1}{2} - iu + 2ig\right)^2}} - \right.\right.$$
$$\left.\left. -\frac{1}{2} + iu - 2ig \right]^{r-1} + h.c. \right\} = \frac{2^{-\frac{5}{2}}}{g^{r-\frac{1}{2}}} I + O\left(\frac{1}{g^r}\right) . \tag{4.10}$$

[5]Here we are again exchanging the above limit (or the corresponding asymptotic series) with the series. This may be troublesome for the next-to-leading correction to $a_{2n}(g)$.

We will go on expanding in the following, but for now we just concentrate on the first term where the integral

$$I = 2\int_{-\infty}^{\infty} \frac{du}{\cosh \pi u}[\sqrt{i+2u} + \sqrt{-i+2u}] \tag{4.11}$$

may be computed by moving the domain of integration on the line Im$u = -1/2$ i.e. changing variable $y = \frac{\pi}{2}(i+2u)$:

$$I = \frac{4\sqrt{2}}{\pi^{\frac{3}{2}}}\int_0^{\infty} dy \frac{\sqrt{y}}{\sinh y} = \frac{2}{\pi}(2^{\frac{3}{2}}-1)\zeta\left(\frac{3}{2}\right) = 3.04084\ldots.. \tag{4.12}$$

As expected, our result, which does not depend on $\kappa$, differs from the numerical computation of [20], valid for $\kappa = 1$: in formula (5.7) of their paper they state that

$$\Delta = L\left[\pi^{\frac{1}{2}} g^{\frac{1}{2}} + O(g^0)\right] , \tag{4.13}$$

which would come out from a value $I = 2\sqrt{2\pi}$. Nevertheless, we may reproduce the 'right' value (4.13) by shifting the accumulation points in the limiting delta-functions, i.e.

$$\rho^{(0)}(u) = \frac{1}{4}\left[\delta\left(u + 2g - \frac{\pi}{4} + \frac{1}{4\pi}\right) + \delta\left(u - 2g + \frac{\pi}{4} - \frac{1}{4\pi}\right)\right] . \tag{4.14}$$

These shifts might allow for a simple interpretation as an effect of sub-leading corrections to $\hat{\rho}(k)$, though the expression before is not the leading term (in the sense that has been defined above).

### 4.2 The case $\kappa = 2$

The $\kappa = 2$ case reveals surprising mathematical simplifications, as here the formulæ above for the density and (the eigenvalues of) the charges ((4.7) and (4.10) respectively) turn out to be the *exact* leading terms without exchanging limit processes. For this reason, we believe this case may be important to understand in a deeper manner the meaning of the strong coupling expansion.[6]

As discussed in the previous subsection, Bethe roots accumulate at $\pm 2g$ independently of the value of $\kappa$. Therefore, their density $\rho(u)$ ought to have the form

$$\rho(u) = \frac{1}{4}\left[f(u + 2g) + f(-u + 2g)\right] , \tag{4.15}$$

where $f(u)$ is a function peaked around $u = 0$, not depending on $g$ and such that

$$\int_{-\infty}^{\infty} du f(u) = 1 . \tag{4.16}$$

Now, we show that the guess (4.15), with a proper choice of $f$, is indeed solution of the linear equation (3.8) when $g \to +\infty$. In fact, this form allow us to compute the charges $Q_r$

[6] This exactness do not exclude, on the contrary may support, the need for changing the expansion as defined by (4.1).

appearing in (3.7). If $r$ is odd, we have $Q_r = 0$, since $\rho(u) = \rho(-u)$. Otherwise, for even $r$, we gain

$$Q_r = \frac{2^{2-r}}{4i^{r-2}(r-1)g^{2r-2}} \left\{ \int_{-\infty}^{\infty} du f(u) \left[ \left( \frac{1}{2} - iu + 2ig \right) \sqrt{1 + \frac{4g^2}{\left(\frac{1}{2} - iu + 2ig\right)^2}} - \right.\right. \tag{4.17}$$
$$\left.\left. -\frac{1}{2} + iu - 2ig \right]^{r-1} + h.c. \right\} = \left[ \frac{2^{-\frac{5}{2}}}{g^{r-\frac{1}{2}}} I_f - (r-1)\frac{2^{-2}}{g^r} + O\left(\frac{1}{g^{r+\frac{1}{2}}}\right) \right],$$

where the leading term is proportional to the f-depending integral

$$I_f = 2 \int_{-\infty}^{\infty} du f(u) [\sqrt{i + 2u} + \sqrt{-i + 2u}] , \tag{4.18}$$

whereas the sub-leading term does not depend on $f$.

Now, we shall plug the result (4.17) into the linear equation (3.8) in which we have consistently developed the coefficients $c_{r,s}(g)$ according to (2.20): in this respect, it is sufficient to insert the leading contribution $c_{r,s}(g) = \delta_{r+1,s}\, g$:

$$\begin{aligned}
\hat\rho(k) = {} & \frac{J_0(2gk)}{2\cosh\frac{k}{2}} - \\
& -4g \sum_{m=1}^{\infty} \frac{J_{2m}(2gk)}{\cosh\frac{k}{2}} \frac{g^{2m+1}}{2(-1)^m} \left[ \frac{2^{-\frac{5}{2}}}{g^{2m+\frac{3}{2}}} I_f - (2m+1)\frac{2^{-2}}{g^{2m+2}} \right] [1 + O(g^{-1})] - \\
& -4g \sum_{m=1}^{\infty} \frac{J_{2m}(2gk)}{\cosh\frac{k}{2}} \frac{g^{2m-1}}{2(-1)^{m-1}} \left[ \frac{2^{-\frac{5}{2}}}{g^{2m-\frac{1}{2}}} I_f - (2m-1)\frac{2^{-2}}{g^{2m}} \right] [1 + O(g^{-1})] .
\end{aligned}$$

Since the leading order terms in the sums cancel out, we are left with the equation

$$\hat\rho(k) = \frac{J_0(2gk)}{2\cosh\frac{k}{2}} - 2\sum_{m=1}^{\infty} (-1)^{m-1} \frac{J_{2m}(2gk)}{2\cosh\frac{k}{2}} [1 + O(g^{-1})] . \tag{4.19}$$

As in the previous subsection, the sum over $m$ is easily performed and eliminates the Bessel functions out of the game:

$$\hat\rho(k) = \frac{\cos(2gk)}{2\cosh\frac{k}{2}} + O(g^{-1}) \Rightarrow \rho(u) = \frac{1}{4}\left[ \frac{1}{\cosh \pi(u+2g)} + \frac{1}{\cosh\pi(u-2g)} \right] + O(g^{-1}) . \tag{4.20}$$

This identifies the function $f(u)$ with

$$f(u) = \frac{1}{\cosh \pi u} , \tag{4.21}$$

furnishing again the leading density at generic $\kappa$ of previous subsection. In particular, the value of the integral (4.18) is still given by (4.12).

As further check, one can verify that the strong coupling limit (4.17) of the charges actually satisfies any equation (3.9), i.e.

$$
\begin{aligned}
Q_r = & \frac{i^{r-2}}{g^{r-1}}\left\{\frac{1}{\pi(r-1)^2\cos\frac{\pi(r-2)}{2}}-\right.\\
& -\frac{8}{\pi}g\sum_{m=1}^{\infty}\frac{(-1)^{m-\frac{r}{2}}}{(2m-1+r)(2m+1-r)}\frac{g^{2m+1}}{2(-1)^m}\left[\frac{2^{-\frac{5}{2}}}{g^{2m+\frac{3}{2}}}I-(2m+1)\frac{2^{-2}}{g^{2m+2}}\right]-\\
& \left.-\frac{8}{\pi}g\sum_{m=1}^{\infty}\frac{(-1)^{m-\frac{r}{2}}}{(2m-1+r)(2m+1-r)}\frac{g^{2m-1}}{2(-1)^{m-1}}\left[\frac{2^{-\frac{5}{2}}}{g^{2m-\frac{1}{2}}}I-(2m-1)\frac{2^{-2}}{g^{2m}}\right]\right\},
\end{aligned}
$$

which is easily verified upon considering the leading order of the l.h.s. $Q_r = O\left(\frac{1}{g^{r-\frac{1}{2}}}\right)$. In conclusion, the expansion (4.17),

$$
\begin{aligned}
Q_r &= \left[\frac{2^{-\frac{5}{2}}}{g^{r-\frac{1}{2}}}I-(r-1)\frac{2^{-2}}{g^r}+O\left(\frac{1}{g^{r+\frac{1}{2}}}\right)\right]=\\
&= \left[\frac{2^{-\frac{3}{2}}}{g^{r-\frac{1}{2}}}\frac{(2^{\frac{3}{2}}-1)}{\pi}\zeta\left(\frac{3}{2}\right)-(r-1)\frac{2^{-2}}{g^r}+O\left(\frac{1}{g^{r+\frac{1}{2}}}\right)\right], \qquad (4.22)
\end{aligned}
$$

is the exact one when $g \to +\infty$ and $\kappa = 2$. Moreover, it also yields the next-to-leading term in the approximation defined in the previous subsection when $\kappa = 1$.

In particular we can estimate the anomalous dimension within the two first orders

$$
\Delta = L(1+2g^2Q_2) = L\left[2^{-\frac{1}{2}}\frac{(2^{\frac{3}{2}}-1)}{\pi}\zeta\left(\frac{3}{2}\right)g^{\frac{1}{2}}+\frac{1}{2}+O(g^{-\frac{1}{2}})\right]. \qquad (4.23)
$$

## 5. Finite size equations

In sections 3 and 4 we have studied the Bethe equations (2.1) and the conserved charges in the anti-ferromagnetic configuration and in the thermodynamic regime. In this section, we want to analyze the finite $L$ case, for slightly more general states, and we also allow for the presence of *holes* in the sequence of real Bethe roots. In this respect, it is convenient to rewrite equations (2.1) in an integral form,– called non-linear integral equation to be distinguished by the linear one of the thermodynamic limit [16]. This non-linear equation has always revealed to be effective for the study of large but finite size corrections, different limit regimes (strong coupling, large size etc.) and for many other issues (cf. for instance [8] and references therein).

As far as $\mathcal{N} = 4$ SYM is concerned, the equations (2.1) at finite $L$ are reliable only up to the order $g^{2L-2}$, because they do not take into account the wrapping effects. Therefore, all the finite $L$ formulæ presented in this section have to be understood as relevant only up to the order $g^{2L-2}$.

As usual [16], we start from equations (2.1) in the logarithmic form (3.1)

$$
iL\ln\frac{X\left(u_k+\frac{i}{2}\right)}{X\left(u_k-\frac{i}{2}\right)} = i\sum_{\substack{j=1\\ j\neq k}}^{M}\ln\frac{u_k-u_j+i}{u_k-u_j-i}-2\sum_{\substack{j=1\\ j\neq k}}^{M}\theta(u_k,u_j)+2\pi K_k\,. \qquad (5.1)
$$

Consequently, we define two functions giving also their different analyticity domain

$$\phi(x) = i\ln\frac{i+x}{i-x} = 2\arctan x\,,\ \ \mathrm{Im}x<1\,; \qquad \Phi(x) = i\ln\frac{X\left(\frac{i}{2}+x\right)}{X\left(\frac{i}{2}-x\right)}\,,\ \ \mathrm{Im}x<1/2\,. \tag{5.2}$$

Upon using the relations

$$\begin{aligned} i\ln\frac{x+i}{x-i} - i\ln\frac{i+x}{i-x} &= -\pi\,, \\ i\ln\frac{X\left(x+\frac{i}{2}\right)}{X\left(x-\frac{i}{2}\right)} - i\ln\frac{X\left(\frac{i}{2}+x\right)}{X\left(\frac{i}{2}-x\right)} &= -\pi\,, \end{aligned}$$

we may recast (5.1) as

$$L\Phi(u_k) = \sum_{j=1}^{M}\phi(u_k-u_j) - 2\sum_{j=1}^{M}\theta(u_k,u_j) + \pi(L-M+1+2K_k)\,. \tag{5.3}$$

Let us define the counting function (analytic in a strip centered on the axis)

$$Z(u) = L\Phi(u) - \sum_{j=1}^{M}\phi(u-u_j) + 2\sum_{j=1}^{M}\theta(u,u_j)\,, \tag{5.4}$$

which renders the Bethe equations in the simple form

$$e^{iZ(u_k)} = -1\,. \tag{5.5}$$

Now, we concentrate on the state characterised by $M$ real Bethe roots, $u_j$, and $H$ holes, $x_h$, which are defined to satisfy (5.5), without being solutions of the original Bethe equations (5.1). In this case, one may express a sum on the Bethe roots for a function $f(u)$ as *logarithmic indicator integral* (for a detailed discussion see [16, 8])

$$\sum_{k=1}^{M} f(u_k) = -\int_{-\infty}^{\infty}\frac{du}{2\pi}\,\frac{d}{du}f(u)\,\left[Z(u)-2L(u)\right] - \sum_{h=1}^{H} f(x_h)\,, \tag{5.6}$$

where we have used the short notation

$$L(u) = \mathrm{Im}\ln\left[1+(-1)^{\delta}e^{iZ(u+i0)}\right]\,, \tag{5.7}$$

and defined $\delta = L-M \mod 2$. In particular, we will be interested in the eigenvalues of the conserved charges

$$\mathcal{Q}_r = \sum_{k=1}^{M} q_r(u_k) = -\int_{-\infty}^{\infty}\frac{du}{2\pi}\,\frac{d}{du}q_r(u)\,\left[Z(u)-2L(u)\right] - \sum_{h=1}^{H} q_r(x_h)\,, \tag{5.8}$$

which scale to the thermodynamic values $Q_r$ (3.3) according to

$$\lim_{L\to\infty}\frac{\mathcal{Q}_r}{L} = Q_r\,. \tag{5.9}$$

Remarkably (5.6) may be applied to (5.4) itself

$$
\begin{aligned}
Z(u) \,=\, & L\Phi(u) - \int_{-\infty}^{\infty} \frac{dv}{2\pi}\frac{2}{(u-v)^2+1}[Z(v)-2L(v)] + \sum_{h=1}^{H}\phi(u-x_h) - \\
& -2\int_{-\infty}^{\infty}\frac{dv}{2\pi}\,\frac{d}{dv}\theta(u,v)\,\,[Z(v)-2L(v)] - 2\sum_{h=1}^{H}\theta(u,x_h)\,.
\end{aligned}
$$

At this stage, it is customary to pass on to the Fourier space and it is possible to go through this step even here. Although it is made rather cumbersome by the involved form of the dressing phase

$$
\theta(u,v) = \kappa\sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty} c_{r,r+1+2\nu}(g)g^{2r+2\nu-1}[q_r(u)q_{r+1+2\nu}(v) - q_{r+1+2\nu}(u)q_r(v)]\,, \tag{5.10}
$$

in Fourier space the following equation shall hold

$$
\begin{aligned}
(1+e^{-|k|})\hat{Z}(k) = & L\frac{2\pi}{i}P\left(\frac{1}{k}\right)e^{-|k|/2}J_0(2gk) + 2e^{-|k|}\hat{L}(k) + \\
& +\sum_{h=1}^{H}e^{-ikx_h}\frac{2\pi}{i}P\left(\frac{1}{k}\right)e^{-|k|} - 2\kappa\int_{-\infty}^{\infty}\frac{dv}{2\pi}\sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty}c_{r,r+1+2\nu}(g)g^{2r+2\nu-1}\,\cdot \\
& \cdot\left[\hat{q}_r(k)\frac{d}{dv}q_{r+1+2\nu}(v) - \hat{q}_{r+1+2\nu}(k)\frac{d}{dv}q_r(v)\right][Z(v)-2L(v)] - \\
& -2\kappa\sum_{h=1}^{H}\sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty}c_{r,r+1+2\nu}(g)g^{2r+2\nu-1}[\hat{q}_r(k)q_{r+1+2\nu}(x_h)-\hat{q}_{r+1+2\nu}(k)q_r(x_h)]\,.
\end{aligned} \tag{5.11}
$$

Inspired by the thermodynamic case, we simplify a little this relation by introducing the charges $\mathcal{Q}_r$

$$
\begin{aligned}
\hat{Z}(k) \,=\, & L\frac{2\pi}{i}P\left(\frac{1}{k}\right)\frac{J_0(2gk)}{2\cosh\frac{k}{2}} + \frac{2}{1+e^{|k|}}\hat{L}(k) + \sum_{h=1}^{H}e^{-ikx_h}\frac{2\pi}{i}P\left(\frac{1}{k}\right)\frac{1}{1+e^{|k|}} + \\
& +\frac{\kappa}{\cosh\frac{k}{2}}\sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty}c_{r,r+1+2\nu}(g)g^{2r+2\nu-1}\left[\frac{2\pi}{g^{r-1}}\frac{1}{i^{r-2}}\frac{J_{r-1}(2gk)}{k}\mathcal{Q}_{r+1+2\nu} - \right. \\
& \left. -\frac{2\pi}{g^{r+2\nu}}\frac{1}{i^{r+2\nu-1}}\frac{J_{r+2\nu}(2gk)}{k}\mathcal{Q}_r\right]\,,
\end{aligned} \tag{5.12}
$$

and also the explicit form (3.4) of the Fourier transform of the charge densities, $\hat{q}_r(k)$. A very crucial difference of this non-linear integral equation from the others in the literature may be stated in the presence of $Z(u)$ in infinite many place, i.e. all the charges $\mathcal{Q}_r$ (5.8).

Concerning the latter, we may correct the thermodynamic system of equations in case of finite $L$. In fact, we first rewrite the expressions (5.6) in terms of Fourier transforms

$$
\begin{aligned}
\mathcal{Q}_s \,=\, & -\int_{-\infty}^{\infty}\frac{dk}{4\pi^2}\hat{q}_s'(-k)[\hat{Z}(k)-2\hat{L}(k)] - \sum_{h=1}^{H}q_s(x_h) = \\
= & \int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{i^{3+s}}{g^{s-1}}\frac{J_{s-1}(2gk)}{e^{\frac{|k|}{2}}}[\hat{Z}(k)-2\hat{L}(k)] - \sum_{h=1}^{H}q_s(x_h)\,.
\end{aligned} \tag{5.13}
$$

Then we insert relation (5.12) into this expression to obtain the corrections

$$\mathcal{Q}_s = \frac{i^{s+2}}{g^{s-1}}\Big[L\int_{-\infty}^{\infty} dk \frac{J_{s-1}(2gk)J_0(2gk)}{k(e^{|k|}+1)} + \\ +2\kappa\sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty} c_{r,r+1+2\nu}(g)(-1)^{1+\nu}\int_{-\infty}^{\infty} dk \frac{J_{s-1}(2gk)J_{r-1}(2gk)}{k(e^{|k|}+1)}\frac{g^{r+2\nu}}{i^{r+2\nu-1}}\mathcal{Q}_{r+2\nu+1} \\ +2\kappa\sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty} c_{r,r+1+2\nu}(g)(-1)^{1+\nu}\int_{-\infty}^{\infty} dk \frac{J_{s-1}(2gk)J_{r+2\nu}(2gk)}{k(e^{|k|}+1)}\frac{g^{r-1}}{i^{r-2}}\mathcal{Q}_r\Big] - \\ -\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{i^{3+s}}{g^{s-1}}\frac{J_{s-1}(2gk)}{\cosh\frac{k}{2}}\hat{L}(k) - \sum_{h=1}^{H}\int_{-\infty}^{\infty}\frac{dk}{2k}e^{-ikx_h}\frac{i^{2+s}}{g^{s-1}}\frac{J_{s-1}(2gk)}{\cosh\frac{k}{2}}\,. \tag{5.14}$$

This relation is exact and, at least in principle, may be efficient in the analysis of the conserved charges, though now $Z(u)$ appears.

## 6. Outlook

In this paper we have pointed out some aspects of the (long-range) asymptotic Bethe equations with dressing factor for the SU(2) sector of planar $\mathcal{N}=4$ SYM. In the specific case of the anti-ferromagnetic state in the thermodynamic limit, we have written an integral equation for the density of Bethe roots and solved it in the leading strong coupling limit. Upon integration, this density gives an approximation for (the eigenvalues of) the commuting charges on this state. Nevertheless, for these charges we have also derived an exact linear system (of algebraic equations): we do think that its investigation may clarify many points, especially about the strong coupling expansion. As a mathematical curiosity, we have observed that these equations are exactly solved in the strong coupling limit if one doubles the dressing factor of [14, 15]. At present, this mathematical simplification has no physical meaning, but the exactness of the solution at this special value of the dressing. As well known in (integrable) statistical field theory, the linear integral equation converts into a non-linear one when the size $L$ becomes finite: in this respect we have shown that this case does not make any exception, though the specific equation is rather involved. Moreover, we have focused our attention at most to the hole 'excitations' on the anti-ferromagnetic configuration and we have derived a finite-size corrected system of equations for the charges. Eventually, we should be able to give simple generalisations of the aforementioned results to the other states/operators along the lines of [16], including *complex pairs.*

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